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Formal Systems, Logic and Semantics coursework

(clausal form)

1) We build the truth table for the formula $((((p \Rightarrow (q \wedge r)) \Rightarrow q) \vee r) \wedge (p \Rightarrow q))$

			(a)	**(b)**	**(c)**	**(d)**	**(e)**	
p	**q**	**r**	**q ∧ r**	**p => a**	**b => q**	**c ∨ r**	**p => q**	**d ∧ e**
0	0	0	0	1	0	0	1	0
0	0	1	0	1	0	1	1	1
0	1	0	0	1	1	1	1	1
0	1	1	1	1	1	1	1	1
1	0	0	0	0	1	1	0	0
1	0	1	0	0	1	1	0	0
1	1	0	0	0	1	1	1	1
1	1	1	1	1	1	1	1	1

We can therefore get the DNF and CNF:

DNF: $(\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge q \wedge r)$

CNF: $\neg(\neg p \wedge \neg q \wedge \neg r) \wedge \neg(p \wedge \neg q \wedge \neg r) \wedge \neg(p \wedge \neg q \wedge r)$
$= (p \vee q \vee r) \wedge (\neg p \vee q \vee r) \wedge (\neg p \vee q \vee \neg r)$

2) We start from the main formula:

$((\exists X)\ a(X,Y) \Rightarrow (\exists X)\ b(X,g(Y)))$

Put it to PNF by moving the quantifiers

$((\exists X)(\exists Z)\ a(X,Y) \Rightarrow b(Z,g(Y)))$

Put it to SF by transforming $\exists$ into $\forall$, using the two evident Skolem Functions

$((\forall W)\ a(x(W),Y) \Rightarrow b(z(W),g(Y)))$

A possible interpretation can simply be made by:

a(X,Y): true if X is 2^Y
b(X,Y): true if X is 2^{Y+1}
g(Y) simply returns Y

The formula therefore becomes "if a number is 2^n, then there exists another number that's equal to 2^{n+1}"

There are no other Skolem forms possible...

3) We start again from the original formula:

$$(\exists Z)((\forall X)(g(Z,X,X) \wedge g(X,Z,X)) \wedge (\forall W)(\exists Y)(g(W,Y,Z) \wedge g(Y,W,Z))$$

And then put it info PNF:

$$(\exists Z)(\forall X)(\forall W)(\exists Y)(g(Z,X,X) \wedge g(X,Z,X) \wedge g(W,Y,Z) \wedge g(Y,W,Z))$$

And then into SF, using some evident Skolem functions:

$$(\forall Q)(\forall X)(\forall W)(g(z(Q),X,X) \wedge g(X,z(Q),X) \wedge g(W,y(X,W),z(Q)) \wedge g(y(X,W),W,z(Q)))$$

We could interpret this formula in multiple ways: one very easy one is to say "g(X,Y,Z) is true", and the formula is satisfied. Yet, this one is too easy!

Second interpretation:

p	**q**	**r**	**g(p,q,r)**
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

To satisfy the formula with this given definition for g(X,Y,Z), we define that the $\exists Z$ is actually Z=1, and y(X,W) is W... Which gives:

$$g(1,X,X) \wedge g(X,1,X) \wedge g(W,W,1) \wedge g(W,W,1) = 1 \wedge 1 \wedge 1 \wedge 1 = 1$$

The formula is therefore satisfied...

4.a) “If Jane had the papers then either they are in the box or someone has stolen them”

Let's reformulate this sentence:

> For all objects, if the object is a paper and Jane had it, then either it is in the box or there exists some object that has stolen it

Let's reformulate the end of it:

> if it is in the box then there exists no object such that that object has stolen it, else there exists some object that has stolen it

Which gives us:

> For all objects, if the object is a paper and Jane had it and it is in the box then there exists no object such that that object has stolen it, and if the object is a paper and Jane had it and it is not in the box then there exists some object that has stolen it

Formula:

$$(\forall X) \quad ((p(X) \wedge j(X) \wedge b(X)) \Rightarrow \neg(\exists Y)\, s(Y,X)) \\ \wedge \quad ((p(X) \wedge j(X) \wedge \neg b(X)) \Rightarrow (\exists Y)\, s(Y,X))$$

Put into PNF:

$$(\forall X)\,(\forall Y)\,(\exists Z) \quad ((p(X) \wedge j(X) \wedge b(X)) \Rightarrow \neg s(Y,X)) \\ \wedge \quad ((p(X) \wedge j(X) \wedge \neg b(X)) \Rightarrow s(Z,X))$$

Put into SF, using evident Skolem functions:

$$(\forall X)\,(\forall Y) \quad ((p(X) \wedge j(X) \wedge b(X)) \Rightarrow \neg s(Y,X)) \\ \wedge \quad ((p(X) \wedge j(X) \wedge \neg b(X)) \Rightarrow s(z(X),X))$$

And, we obtain the Clausal form:

$$(((p(X) \wedge j(X) \wedge b(X)) \Rightarrow \neg s(Y,X)) \wedge ((p(X) \wedge j(X) \wedge \neg b(X)) \Rightarrow s(z(X),X)))$$

4.b) The greatest of all detectives played the violin.

Let's reformulate this sentence:

> For all objects, if the object is the greatest of all detectives then the object played the violin.

Formula:

$$(\forall X)\, gd(X) \Rightarrow v(X)$$

This formula is already in SF... And, to put it into Clausal form, it's easy:

$$gd(X) \Rightarrow v(X)$$

4.c)Some people do not listen to anything other people say, except for flattery

Let's reformulate this sentence:

> There exists some objects such that those object are people and for all other objects that are people, if those second objects tell something to the first ones, the first will listen to it if and only if it is a flattery.

Formula:

$$(\exists X)\ (\forall Y)\ ((\neg(X=Y) \wedge p(X) \wedge p(Y) \wedge s(Y,X,Z)) \Rightarrow (l(X,Z) \Leftrightarrow f(Z))$$

This formula is already in PNF... Let's put it into SF, using evident Skolem functions:

$$(\forall X)\ (\forall Y)\ ((\neg(x(X)=Y) \wedge p(x(X)) \wedge p(Y) \wedge s(Y,x(X),Z)) \Rightarrow (l(x(X),Z) \Leftrightarrow f(Z))$$

We now erase the quantifiers and start developing the formula:

$$\begin{array}{ll} & \neg(\neg(x(X)=Y) \wedge p(x(X)) \wedge p(Y) \wedge s(Y,x(X),Z)) \vee (l(x(X),Z) \wedge f(Z)) \\ = & (x(X)=Y) \vee \neg p(x(X)) \wedge \neg p(Y) \wedge \neg s(Y,x(X),Z) \vee (l(x(X),Z) \wedge f(Z)) \\ = & \neg(X=Y) \wedge p(x(X)) \wedge p(Y) \wedge s(Y,x(X),Z) \wedge \neg(\neg l(x(X),Z) \vee \neg f(Z))) \\ = & \neg(X=Y) \wedge p(x(X)) \wedge p(Y) \wedge s(Y,x(X),Z) \wedge l(x(X),Z) \wedge f(Z) \end{array}$$

The last line is indeed in some Clausal form...

4.d) Every flock in the hen hatched on Tuesday and every egg hatched on Friday.

Let's reformulate this sentence:

> For all objects, if the object is a flock and it is in the hen then it has laid one and only one egg on Tuesday and that egg hatched on Friday.

Formula:

$$(\forall X)\ (f(X) \wedge h(X)) \Rightarrow ((\exists Y)\ lt(X,Y) \wedge (lt(X,Z) \Rightarrow (Y=Z)) \wedge hf(Y))$$

Put into PNF:

$$(\forall X)\ (\exists Y)\ (f(X) \wedge h(X)) \Rightarrow (lt(X,Y) \wedge (lt(X,Z) \Rightarrow (Y=Z)) \wedge hf(Y))$$

Put into SF, using the evident Skolem function y(X):

$$(\forall X)\ (f(X) \wedge h(X)) \Rightarrow (lt(X,y(X)) \wedge (lt(X,Z) \Rightarrow (y(X)=Z)) \wedge hf(y(X)))$$

We now erase the quantifiers and start developing the formula:

$$\begin{array}{ll} & \neg(f(X) \wedge h(X)) \vee (lt(X,y(X)) \wedge (\neg lt(X,Z) \vee (y(X)=Z)) \wedge hf(y(X))) \\ = & \neg f(X) \vee \neg h(X) \vee (lt(X,y(X)) \wedge hf(y(X)) \wedge \neg lt(X,Z)) \vee (lt(X,y(X)) \wedge hf(y(X)) \wedge (y(X)=Z))) \\ = & f(X) \wedge h(X) \wedge (\neg lt(X,y(X)) \vee \neg hf(y(X)) \vee lt(X,Z))) \wedge (\neg lt(X,y(X)) \vee \neg hf(y(X)) \vee \neg y(X)=Z))) \end{array}$$

... I stop !

5.a) $\Gamma \models A$ means "if an interpretation I makes Γ true, then it also makes A true". Therefore:

If $\Gamma \models A$ then there is no interpretation such that $\Gamma \wedge \neg A$

Therefore $\Gamma \models A \Rightarrow (\Gamma, \neg A)$ is not satisfiable... (1)

$(\Gamma, \neg A)$ not satisfiable means that there is no interpretation such that Γ is true and A is false. Therefore, $\Gamma \Rightarrow A$ is also true.

$\Gamma \models A$ means that if Γ is true for some interpretation then $\Gamma \wedge A$ is also true... So, $\Gamma \Rightarrow \Gamma \wedge A$.

Let's analyze:

$\Gamma \Rightarrow A \Leftrightarrow \neg\Gamma \vee A$
$\Gamma \Rightarrow \Gamma \wedge A \Leftrightarrow \neg\Gamma \vee (\Gamma \wedge A)$

Γ	**A**	$\neg\Gamma \vee$ **A**	$\neg\Gamma \vee (\Gamma \wedge$ **A**$)$
0	0	1	1
0	1	1	1
1	0	0	0
1	1	1	1

We see that the two formulas are equivalent, therefore have $(\Gamma, \neg A)$ not satisfiable $\Rightarrow \Gamma \models A$ (2)

(1) + (2) makes: $\Gamma \models A \Leftrightarrow (\Gamma, \neg A)$ not satisfiable

5.b) Γ is a set of clauses, and therefore can be written as a conjunction of disjunctions (or vice-versa); we therefore can see it as a function with n arguments (n >= 0)

"Γ is satisfiable" means that for at least one set of arguments $x_1 \ldots x_n$, $\Gamma(x_1 \ldots x_n)$ is true.
(if n=0, it simply means that Γ is true)

5.c) We've proved that $\Gamma \models A \Leftrightarrow (\Gamma, \neg A)$ not satisfiable ; and also that Γ is satisfiable <=> there exists at least one set of arguments $x_1 \ldots x_n$ such that $\Gamma(x_1 \ldots x_n)$ is true.

Based on this idea, we could suppose that to see for which combination of variables Γ is satisfiable we can simply build a table with all possible combinations of x_i and calculate Γ for those values...

This method sure has multiple disadvantages:

- it is quite an easy computer program, but will generate m^n operations! (m being the dimension of the space, 2 for (true,false), 3 for (r,g,b), etc.)
- it sure does not mean much when the space we work on is infinite, such as R or C